Hill-type formula and Krein-type trace formula for S-periodic solutions in ODEs

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Dedicate to Rou-Huai Wang's 90th birth anniversary

Abstract

The present paper is devoted to studying the Hill-type formula and Krein-type trace formula for ODE, which is a continuous work of our previous work for Hamiltonian systems [4]. Hill-type formula and Krein-type trace formula are given by Hill at 1877 and Krein in 1950's separately. Recently, we find that there is a closed relationship between them [4]. In this paper, we will obtain the Hill-type formula for the S-periodic orbits of the first order ODEs. Such a kind of orbits is considered naturally to study the symmetric periodic and quasi-periodic solutions. By some similar idea in [4], based on the Hill-type formula, we will build up the Krein-type trace formula for the first order ODEs, which can be seen as a non-self-adjoint version of the case of Hamiltonian system.

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Key Words. Hill-type formula; Krein-type trace formula ; Fredholm determinant; Hilbert-Schmidt operator

1 Introduction

In the present paper, we will study the Hill-type formula and Krein-type trace formula for S-periodic solutions for the first order ODE. Hill-type formula was introduced by Hill [2] when he considered the motion of lunar perigee at 1877, and the Krein-type trace formula was built up by Krein [6, 7] in 1950's when he studied the stability of Hamiltonian systems. Although they appeared separately, there is a closed relationship between them. In fact, the Krein-type trace formula could be derived by the Hill-type formula for S-periodic orbits in Hamiltonian systems. Moreover, motivated by the Krein's original work, the Krein-type trace formula was used to study the stability problem in n-body problem, details could be found in [5, 4]. In the case of Hamiltonian

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system, the corresponding differential operators are self-adjoint, and the formulas for the first order ODE can be seen as a non-self-adjoint version.

Let $M_m(\mathbb{C})$ be the set of $m\times m$ matrices on $\mathbb{C}^m$, and denote by

$$\mathfrak{B}(m) = C([0,T], M_m(C)), \tag{1.1}$$

the set of continuous path of $m\times m$ matrices on $[0,T]$. We consider the n-dimensional first order ODE with the S-periodic boundary problem

$$\begin{aligned} \dot{x}(t) &= D(t)x(t), && (1.2)\\ x(0) &= Sx(T), && (1.3) \end{aligned}$$

where S is an orthogonal $n\times n$ matrix and $D\in\mathfrak{B}(n)$. It is natural to study the S-periodic solutions of the first order ODE when we study the symmetric periodic solutions and quasi-periodic solutions. In what follows, we always denote by $\gamma_D(t)$ the fundamental solution of the first order ODE (1.2), that is, $\dot{\gamma}_D(t) = D(t)\gamma_D(t)$ with $\gamma_D(0) = I_n$.

Consider $\frac{d}{dt}$ as the unbounded closed operator densely defined on $L^2(0,T;\mathbb{R}^n)$ with the domain

$$D_S = \Big\{ z(t)\in W^{1,2}([0,T],\mathbb{C}^n) \,\big|\, z(0) = Sz(T)\Big\}.$$

D is a bounded operator acting on $L^2(0,T;\mathbb{R}^n)$ defined by $(Dz)(t) = D(t)z(t)$. In this paper, we will prove the following Hill-type formula.

Theorem 1.1. *Let $D\in\mathfrak{B}(n)$ and S be an orthogonal matrix. Then, for any $\nu\in\mathbb{C}$*

$$\det\Big[\Big(\frac{d}{dt} - D + \nu I_n\Big)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\Big] = (-1)^n|C(S)|e^{-\frac{n\nu T}{2}}e^{-\frac{1}{2}\int_0^T Tr(D)dt}\det(S\gamma_D(T) - e^{\nu T}I_n), \tag{1.4}$$

where $\hat{P}_0$ is the orthogonal projection onto $\ker(S-I_n)$ and $C(S)$ is a constant depending only on S.

Here, if we let $W = \ker(S-I_n)^\perp$ and $k_0 = \dim\ker(S-I_n)$, then $C(S) = T^{-k_0}\frac{1}{\det[(S-I_n)|_W]}$.

Remark 1.2. *1) In Theorem 1.1, since $(D+\hat{P}_0-\nu I_n)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}$ is not a trace class operator, but a Hilbert-Schmidt operator. Hence, the infinite determinant*

$$\det\Big[\Big(\frac{d}{dt} - D + \nu I_n\Big)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\Big] = \det\Big[id - (D+\hat{P}_0-\nu I_n)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\Big]$$

is not the classical Fredholm determinant. In fact, we can define the determinant in the following way. Let $\hat{P}_N$ be the orthogonal projections onto

$$V_N = \bigoplus_{\nu\in\sigma(\frac{d}{dt}),|\nu|\le N} \ker\Big(\nu - \frac{d}{dt}\Big).$$

And the conditional Fredholm determinant

$$\det\Big[\Big(\frac{d}{dt} - D + \nu I_n\Big)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\Big] = \lim_{N\to\infty}\det\Big[id - \hat{P}_N(D+\hat{P}_0-\nu I_n)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\hat{P}_N\Big].$$

2) If $S = I_n$, then the boundary condition problem (1.2-1.3) is the canonical periodic boundary condition problem. In this case $C(S) = T^{-n}$. When the period $T = 1$, the Hill-type formula was obtained by Denk [1]. Based on the Hill-tpye formula, Denk developed some efficient numerical method for the ODEs.

Similar to [4], the Hill-type formula (1.4) is the starting point of the Krein-type trace formula. To get the trace formula, for $D_0(t), D(t) \in \mathfrak{B}(n)$, we consider the eigenvalue problem

$$\dot{z}(t) = (D_0(t) + \alpha D(t))z(t), \tag{1.5}$$

$$z(0) = Sz(T), \tag{1.6}$$

that is, to find the $\alpha \in \mathbb{C}$ such that the system (1.5-1.6) has a nontrivial solution. As above, let γ_α be the fundamental solution of (1.5). To state the trace formula, we need some notations. Write $M = S\gamma_0(T)$ and $\hat{D}(t) = \gamma_0^{-1}(t)D(t)\gamma_0(t)$. For $k \in \mathbb{N}$, let

$$M_k = \int_0^T \hat{D}(t_1) \int_0^{t_1} \hat{D}(t_2) \cdots \int_0^{t_{k-1}} \hat{D}(t_k) dt_k \cdots dt_2 dt_1,$$

and

$$G_k = M_k M \left(M - e^{\nu T} I_{2n}\right)^{-1}.$$

Theorem 1.3. *Let $\nu \in \mathbb{C}$ such that $\frac{d}{dt} - D_0 - \nu$ is invertible, $F = D\left(\frac{d}{dt} - D_0 + \nu I_n\right)^{-1}$, then*

$$Tr(F) = \frac{1}{2}\int_0^T Tr(D(t))dt - Tr(G_1), \tag{1.7}$$

and for any positive integer $m \geq 2$,

$$Tr(F^m) = m\sum_{k=1}^{m} \frac{(-1)^k}{k}\Big[\sum_{j_1+\cdots+j_k=m} Tr(G_{j_1}\cdots G_{j_k})\Big]. \tag{1.8}$$

Remark 1.4. *(1). For $m = 1$, F is not a trace class operator but a Hilbert-Schmidt operator. And hence $Tr(F)$ is not the usual trace but a kind of conditional trace[5]. That is*

$$Tr(F) = \lim_{N\to\infty} Tr\hat{P}_N F \hat{P}_N. \tag{1.9}$$

For $m \geq 2$, F^m are trace class operators. Obviously, for $\nu = 0$, λ_i is the eigenvalues of (1.5)-(1.6), if and only if $\frac{1}{\lambda_j}$ is an eigenvalue of F.

$$\sum_j \frac{1}{\lambda_j^m} = m\sum_{k=1}^{m} \frac{(-1)^k}{k}\Big[\sum_{j_1+\cdots+j_k=m} Tr(G_{j_1}\cdots G_{j_k})\Big],$$

where the sum takes for eigenvalues counting algebraic multiplicity.

(2). The trace formula (1.7) for ODE is different from that for Hamiltonian system, and the reason is that the differential operator here is not self-adjoint any more.

Remark 1.5. *The idea and the techniques are similar to those in [4], however, there are at least two reasons to write this paper. For the first, it is not like that Hamiltonian system comes from mechanic system mostly, the ODE systems come from many areas, then the Hill-type trace formula and Krein-type trace formula for ODE will be more convenient to be used. For the second, since the differential operators for ODEs are not self-adjoint, we should do some spectral analysis for ODE carefully. Although the formulas for ODE are similar to that for Hamiltonian system in [4], however, it is difficult to deduce them.*

The trace formula for Hamiltonian systems is a useful tool in study the stability of Hamiltonian systems, by using the trace formula and Maslov-type index theory [8], some applications to the n-body problem is given in [4, 3]. In Section 5, as an application, we will give a generalization of Krein's work on second order systems [6].

This paper is organized as follows, in section 2, we review the basic properties of conditional Fredholm determinant and conditional trace. In section 3, we derive the Hill-type formula for the S-periodic orbits in ODE. In section 4, we get the trace formula from the Hill-type formula. Finally, as an example, we will reformulate Krein's trace formula from our viewpoint.

2 Preliminaries

In this section, we will mainly recall some fundamental properties of conditional Fredholm determinant, which was developed in [5]. In the classical settings, the Fredholm determinant $\det(id+F)$ is defined for a trace class operator F, details could be found in [10]. However, when we study the ODEs, the operators we encountered are of the form $(id+F)$ with that F is not a trace class operator, but a Hilbert-Schmidt operator. Thus, the Fredholm determinant $\det(id+F)$ could not be defined, and the conditional Fredholm determinant will be used instead. To define the conditional Fredholm determinant , the trace finite condition plays an important role.

Let $\{P_k\}$ be a sequence of finite rank projections, such that the following conditions are satisfied,

(1) for $k \leq m$, $Range(P_k) \subseteq Range(P_m)$,

(2) P_k converges to id in the strong operator topology.

A Hilbert-Schmidt operator F is called to have the trace finite condition with respect to P_k, if the limit $\lim_{k\to\infty} Tr(P_kFP_k)$ exists, and the limit is finite. Clearly, a trace class operator has the trace finite condition. Obviously, all the Hilbert-Schmidt operators with trace finite conditions consists a linear space, that is, if F_1 and F_2 are Hilbert-Schmidt operators with trace finite condition, then $\alpha_1F_1+\alpha_2F_2$ has the trace finite condition.

By [10], if F is a Hilbert-Schmidt operator, then the regularized Fredholm determinant is defined by

$$\det{}_2(id+F) = \det\Big((id+F)e^{-F}\Big). \tag{2.1}$$

As been pointed in [5], if F is a Hilbert-Schmidt operator with trace finite condition, then the conditional Fredholm determinant can be defined by

$$\begin{aligned}\det(id+F) &= \lim_{k\to\infty}\det(id+P_kFP_k)\\ &= \det{}_2(id+F)\lim_{k\to\infty} e^{Tr(P_kFP_k)},\end{aligned} \tag{2.2}$$

where, P_kFP_k are finite rank operators, and hence $\det(id + P_kFP_k)$ is well defined. As we proved in [5], many fundamental properties of conditional Fredholm determinant are similar to that of the usual Fredholm determinant.

Proposition 2.1. *1) If F_1 and F_2 are Hilbert-Schmidt operators with trace finite condition, then*

$$\det\left((id + F_1)(id + F_2)\right) = \det(id + F_1)\det(id + F_2).$$

2) Let $E = E_1 \oplus E_2$, and F_i be Hilbert-Schmidt operators on E_i with trace finite condition with respect to $P_k^{(i)}$, $i = 1, 2$. Let $F = F_1 \oplus F_2$, then F has the trace finite condition with respect to $P_k^{(1)} \oplus P_k^{(2)}$, and

$$\det(id + F) = \det(id_{E_1} + F_1)\det(id_{E_2} + F_2),$$

where id_{E_i} are identities on E_i, for $i = 1, 2$.

Similar to that we have given in [5], it is not hard to show that $\det(id + \alpha F)$ is analytic on α, for a Hilbert-Schmidt operator F with trace finite condition. For reader's convenience, we will give the proof of it.

Lemma 2.2. *Let F be a Hilbert-Schmidt operator with trace finite condition with respect to $\{P_k\}$, then $\det(id + \alpha F)$ is an entire function.*

Proof. Write

$$f_k(\alpha) = \det(id + \alpha P_kFP_k), \tag{2.3}$$

then, by the definition, f_k converges to $f(\alpha) = \det(id + \alpha P_kFP_k)$ point-wisely. Moreover, since P_k are finite rank projections, P_kFP_k are finite rank operators, and hence they are trace class. It follows that $f_k(\alpha)$ are entire functions. By Montel's Theorem, it suffices to show that, $\{f_k\}$ is locally bounded, that is, for any compact set $K \subseteq \mathbb{C}$, there is a constant $C > 0$ depending only on K such that

$$\sup\left\{|f_k(\alpha)| \,\middle|\, \alpha \in K\right\} < C.$$

By [5, Lemma 2.3],

$$f_k(\alpha) = \det{}_2(id + \alpha P_kFP_k)e^{-\alpha TrP_kFP_k}.$$

Firstly, by [10, Theorem 9.2],

$$\sup_{\alpha\in K}|\det{}_2(id + \alpha P_kFP_k)| \le \sup_{\alpha\in K} e^{C||\alpha P_kFP_k||_2^2} \le e^{C||F||_2^2 \sup\limits_{\alpha\in K}|\alpha|},$$

where $||\cdot||_2$ is the Hilbert-Schmidt norm. Secondly, since $\lim\limits_{N\to\infty} TrP_NFP_N$ exists, we have that, there is a constant $C_1 > 0$ such that for any $k \in \mathbb{N}$, $|TrP_kFP_k| < C_1$. It follows that

$$\sup_{\alpha\in K}|e^{-\alpha TrP_kFP_k}| \le e^{C_1 \sup\limits_{\alpha\in K}|\alpha|}.$$

Therefore,

$$\sup\left\{|f_k(\alpha)| \,\middle|\, \alpha \in K\right\} < C$$

for some constant C which depends only on K, that is, $\{f_k\}$ is locally bounded. The proof is complete. □

Remark 2.3. *In the proof of Lemma 2.2, we show that* $\{\det(id+\alpha P_k F P_k)\}$ *is a normal family, and hence there is a subsequence of* $\{\det(id+\alpha P_k F P_k)\}$*, say* $\{\det(id+\alpha P_{k_j} F P_{k_j})\}$*, which is convergent uniformly to* $\det(id+\alpha F)$ *on any compact subset of* $\mathbb{C}$*. Denote by*

$$g_{k_j}(\alpha) = \det(id + \alpha P_{k_j} F P_{k_j}).$$

Following [10], we call a function $f : X \to Y$ between Banach spaces, finitely analytic if and only if, for all $A_1, ..., A_n \in X$, $f(z_1 A_1 + ... + z_n A_n)$ is an entire function of $z_1, ..., z_n$ from $\mathbb{C}^n$ to Y. This concept is very useful in studying the conditional Fredholm determinant. An important property is the following:

Theorem 2.4. *[10, pp.45, Theorem5.1]. If a finitely analytic function* f *satisfied* $f(x) \le G(\| x \|)$ *for some monotone function* G *on* $[0, \infty)$*, then* f *is Fréchet differentiable for all* $x \in X$*, and* Df *is finitely analytic function from* X *to* $\mathfrak{L}(X, Y)$ *(Banach space of the linear operators from* X *to* Y*), and*

$$(Df)(x) \le G(\| x \| + 1).$$

Next, we will consider the Taylor expansion of $\det(id + \alpha F)$. Write

$$g(\alpha) = \det(id + \alpha F). \tag{2.4}$$

By [10, Theorem 5.4], we have the following lemma.

Lemma 2.5. *Let* $g_k(\alpha) = \det(id + \alpha F_k)$*. Then the Taylor expansion near* 0 *for* $g_k(\alpha)$ *is*

$$g_k(\alpha) = \sum_{m=0}^{\infty} \alpha^m a_{k,m}/m!,$$

where

$$a_{k,m} = \det \begin{pmatrix} TrF_k & m-1 & 0 & \cdots & 0 \\ Tr(F_k^2) & TrF_k & m-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Tr(F_k^{m-1}) & Tr(F_k^{m-2}) & \cdots & TrF_k & 1 \\ Tr(F_k^m) & Tr(F_k^{m-1}) & \cdots & Tr(F_k^2) & TrF_k \end{pmatrix}.$$

and $F_k = P_k F P_k$*.*

The following reasoning is similar to that in [4, Section 2], and we write here again for reader's convenience.

Now, let $g(\alpha) = \sum\limits_m \frac{a_m}{m!} \alpha^m$ be the Taylor expansion of $g(\alpha)$. Since $g_{k_j}(\alpha)$ converges to $g(\alpha)$ on any compact subset of $\mathbb{C}$, the coefficients $a_{k_j,m} \to a_m$. Notice that F is a Hilbert-Schmidt operator with trace finite condition, then $TrF_k \to TrF$. Therefore

$$a_m = \det \begin{pmatrix} TrF & m-1 & \cdots & 0 \\ Tr(F^2) & TrF & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Tr(F^m) & Tr(F^{m-1}) & \cdots & TrF \end{pmatrix},$$

Note that for α small, by [10, p.47,(5.12)], we have that

$$\det(id+\alpha F_k)=\exp\Big(\sum_{m=1}\frac{(-1)^{m+1}}{m}\alpha^m Tr(F_k^m)\Big). \tag{2.5}$$

By taking limit, we have the following theorem.

Theorem 2.6. *Let $g(\alpha)=\det(id+\alpha F)$. Then for α small,*

$$g(\alpha)=\exp\Big(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\alpha^m Tr(F^m)\Big). \tag{2.6}$$

At last of this section, we will brief review the Hill-type formula for Hamiltonian systems. Assume $\bar{S}$ be a symplectic orthogonal matrix, for $\bar{S}$-periodic orbits in Hamiltonian system, the Hill-type formula was proved in [5], which is listed as follows.

Theorem 2.7. *There is a constant $C(\bar{S})>0$, which depends only on $\bar{S}$, such that for any $\nu\in\mathbb{C}$*

$$\det\Big(\Big(-J\frac{d}{dt}-B-\nu J\Big)\Big(-J\frac{d}{dt}+P_0\Big)^{-1}\Big)=C(\bar{S})e^{-\frac{1}{2}\int_0^T Tr(JB(t))dt}\lambda^{-n}\det(\bar{S}\gamma(T)-\lambda I_{2n}), \tag{2.7}$$

where $\lambda=e^{\nu T}$ and $C(\bar{S})>0$ is a constant depending only on $\bar{S}$.

Precisely,

$$C(\bar{S})=2^{-n}(2/T^2)^{k_0}\prod_{j=k_0+1}^{n}\frac{1}{1-\cosh(\sqrt{-1}\nu_j T)},$$

where $k_0=\dim\ker(R-\sqrt{-1}Q-I_n)$. In fact, if we denote $\bar{W}=\ker(\bar{S}-I_{2n})^{\perp}$, then

$$C(\bar{S})=T^{-2k_0}\frac{1}{\det(\bar{S}-I_{2n})|_{\bar{W}}}.$$

Obviously, $C(\bar{S})>0$. If $\bar{S}=I_{2n}$, i.e. for the periodic boundary conditions, $C(\bar{S})=T^{-2n}$, and if $\ker(\bar{S}-I_{2n})=0$, then

$$C(\bar{S})=\det(\bar{S}-I_{2n})^{-1}.$$

For a $n\times n$ orthogonal matrix S, let $W=\ker(S-I_n)^{\perp}$, and we also denote

$$C(S)=T^{-k_0}\frac{1}{\det(S-I_n)|_W},$$

where $k_0=\dim\ker(S-I_n)$. It can be shown that $C(S)=(-1)^{\sigma}|C(S)|$, where σ is the orientation of S. Then in the special case $\bar{S}=\begin{pmatrix} S & 0_n \\ 0_n & S\end{pmatrix}$, where S is a $n\times n$ orthogonal matrix,

$$C(\bar{S})=C(S)^2. \tag{2.8}$$

3 Hill-type formula for S-periodic solutions of ODEs

In this section, we will mainly derive the Hill-type formula for the first order ODE.

3.1 Hill-type formula for the first order ODEs

To derive the Hill-type formula, we will consider the conditional Fredholm determinant of ODEs. Firstly, we consider the first order ODE with the S-periodic boundary condition,

$$\dot{z}(t) = D(t)z(t) \tag{3.1}$$

$$z(0) = Sz(T), \tag{3.2}$$

where S is an orthogonal matrix on $\mathbb{R}^n$ and $D \in \mathfrak{B}(n)$. To continue, we will consider the spectrum of $\frac{d}{dt}$ with the S-boundary condition. Denote the domain of $\frac{d}{dt}$ by

$$D_S = \Big\{ z(t) \in W^{1,2}([0,T], \mathbb{C}^n) \,\big|\, z(0) = Sz(T) \Big\}.$$

Obviously $\sqrt{-1}\frac{d}{dt}$ is a self-adjoint operator on $E = L^2([0,T], \mathbb{C}^{2n})$ with domain D_S.

For simplicity, write $A = \sqrt{-1}\frac{d}{dt}$. By some simple calculation, we have that $\nu \in \sigma(A)$ if and only if $\ker(Se^{\sqrt{-1}\nu I_n T} - I_n)$ is nontrivial, and the corresponding eigenvector is $e^{\nu\sqrt{-1}I_n t}\xi$, where $\xi \in \ker(Se^{\sqrt{-1}\nu I_n T} - I_n)$. Since S is an orthogonal matrix, there is a unitary matrix U such that

$$U^*SU = \begin{pmatrix} e^{-\sqrt{-1}\theta_1} & & & \\ & e^{-\sqrt{-1}\theta_2} & & \\ & & \ddots & \\ & & & e^{-\sqrt{-1}\theta_n} \end{pmatrix}.$$

where $0 = \theta_1 = \cdots = \theta_{k_0} < \theta_{k_0+1} \le \cdots \le \theta_n < 2\pi$. It is easy to check that $\nu \in \sigma(A)$ if and only if $\nu = (\theta_j + 2k\pi)/T$, for some $1 \le j \le n$; furthermore, $(U^*SUe^{\sqrt{-1}\nu_j I_n T} - I_n)\xi_j = 0$ if and only if $\xi_j = (\underbrace{0, 0, \cdots, 0}_{j-1}, 1, 0, \cdots, 0)$. We have the following lemma.

Lemma 3.1. *The spectrum of $\frac{d}{dt}$ (with the S-boundary condition) is periodic with the period of $2\pi/T$. Precisely, let $\nu_j = \theta_j/T$, then*

$$\sigma\Big(\frac{d}{dt}\Big) = \bigcup_{j=1}^{n} \big\{ -\sqrt{-1}(\nu_j + 2k\pi/T) \,\big|\, k \in \mathbb{Z} \big\}.$$

In fact $\nu_j = \theta_j/T$. Moreover, let $-\sqrt{-1}(\nu_j + 2k\pi/T) \in \sigma(\frac{d}{dt})$, then the corresponding eigenvector is $e^{(\nu_j + 2k\pi/T)\sqrt{-1}I_n t}U\xi_j$, where $\xi_j = (\underbrace{0, 0, \cdots, 0}_{j-1}, 1, 0, \cdots, 0)$.

It is worth being pointed out that

$$0 = \nu_1 = \cdots = \nu_{k_0} < \nu_{k_0+1} \le \cdots \le \nu_n < 2\pi/T, \tag{3.3}$$

where $k_0 = \dim\ker(S - I_n)$.

Remark 3.2. *Since S is an orthogonal matrix, $e^{i\theta}$ is an eigenvalue of S if and only if $e^{-i\theta}$ is an eigenvalue of S. By the argument before Lemma 3.1 we know that, under the S-boundary condition, the spectrum*

$$\sigma\Big(\frac{d}{dt}\Big) = \sigma\Big(-\frac{d}{dt}\Big), \tag{3.4}$$

which will be used later.

By Lemma 3.1, $\left(\frac{d}{dt}+\hat{P}_0\right)^{-1}$ is a Hilbert-Schmidt operator, where $\hat{P}_0$ is the orthogonal projections onto $\ker\left(\frac{d}{dt}\right)$. Applying a similar reasoning to [5, Remark 2.9] shows that $(D+\hat{P}_0)(\frac{d}{dt}+\hat{P}_0)^{-1}$ has the trace finite condition with respect to $\{\hat{P}_N\}$, where $\hat{P}_N$ are orthogonal projections onto

$$V_N = \bigoplus_{\nu\in\sigma(\frac{d}{dt}),|\nu|\le N} \ker\left(\nu - \frac{d}{dt}\right). \tag{3.5}$$

In fact, we have the following lemma. For reader's convenience, we will give the proof of it.

Lemma 3.3. *Under the above assumption, we have that*

$$\lim_{N\to\infty} Tr\Big(\hat{P}_N D\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\hat{P}_N\Big) = \frac{1}{T}\sum_{j=1}^{k_0}\int_0^T \hat{D}_{jj}dt + \sqrt{-1}\sum_{j=k_0+1}^{n}\frac{1+\cos T\nu_j}{2\sin T\nu_j}\int_0^T \hat{D}_{jj}(t)dt,$$

where $\hat{D}=U^*DU$.

Proof. For $1\le j\le n$, let

$$M_j = \bigoplus_{k\in\mathbb{Z}} \ker\left(\frac{d}{dt}+\sqrt{-1}\Big(\nu_j+\frac{2k\pi}{T}\Big)\right), \tag{3.6}$$

then $E=\oplus M_j$. Let $\frac{1}{\sqrt{T}}e^{(\nu_j+2k\pi/T)\sqrt{-1}I_n t}U\xi_j$ be the eigenvector for $-\frac{d}{dt}$ with respect to the eigenvalue $-\sqrt{-1}\Big(\nu_j+\frac{2k\pi}{T}\Big)$. Then

$$\begin{aligned}
&\lim_{N\to\infty} Tr\Big(\hat{P}_N D\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\hat{P}_N\Big)\\
&=\sum_{j=1}^{n}\lim_{N\to\infty}\sum_{|k|<N}\Big\langle D\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\frac{1}{\sqrt{T}}e^{(\nu_j+2k\pi/T)\sqrt{-1}I_n t}U\xi_j, \frac{1}{\sqrt{T}}e^{(\nu_j+2k\pi/T)\sqrt{-1}I_n t}U\xi_j\Big\rangle.
\end{aligned}$$

For $1\le j\le k_0$,

$$\lim_{N\to\infty}\sum_{|k|<N}\Big\langle D\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\frac{1}{\sqrt{T}}e^{(\nu_j+2k\pi/T)\sqrt{-1}I_n t}U\xi_j, \frac{1}{\sqrt{T}}e^{(\nu_j+2k\pi/T)\sqrt{-1}I_n t}U\xi_j\Big\rangle = \frac{1}{T}\int_0^T \hat{D}_{jj}(t)dt.$$

For $k_0+1\le j\le n$,

$$\begin{aligned}
&\lim_{N\to\infty}\sum_{|k|<N}\Big\langle D\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}\frac{1}{\sqrt{T}}e^{(\nu_j+2k\pi/T)\sqrt{-1}I_n t}U\xi_j, \frac{1}{\sqrt{T}}e^{(\nu_j+2k\pi/T)\sqrt{-1}I_n t}U\xi_j\Big\rangle\\
&= 2\lim_{N\to\infty}\sum_{k\le N} -\frac{1}{\sqrt{-1}(\nu_j+2k\pi/T)}\int_0^T \hat{D}_{jj}(t)dt\\
&= \sqrt{-1}\frac{1+\cos T\nu_j}{\sin T\nu_j}\int_0^T \hat{D}_{jj}(t)dt.
\end{aligned}$$

The above calculations imply the desired result. ∎

Now, we will consider the conditional Fredholm determinant $\det\Big((\frac{d}{dt}-D)(\frac{d}{dt}+\hat{P}_0)^{-1}\Big)$ with respect to $\hat{P}_N$. As we have done before, write

$$\Big(\frac{d}{dt}-D\Big)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}=id-(D+\hat{P}_0)\Big(\frac{d}{dt}+\hat{P}_0\Big)^{-1}.$$

Hence, the conditional Fredholm determinant $\det\Big(id-(D+\hat{P}_0)(\frac{d}{dt}+\hat{P}_0)^{-1}\Big)$ is well-defined. Please note that for any $D_1\in\mathfrak{B}(n)$ such that $\frac{d}{dt}+D_1$ is invertible, the operator $D(\frac{d}{dt}+D_1)^{-1}$ is a Hilbert-Schmidt operator with the trace finite condition with respect to $\hat{P}_N$. Therefore the infinite determinant $\det\left[\Big(\frac{d}{dt}-D\Big)\Big(\frac{d}{dt}+D_1\Big)^{-1}\right]$ is well defined.

In this remaining part of this subsection, we will deduce the Hill-type formula for first order ODE with S-periodic boundary condition, where S is an orthogonal matrix. Consider the following equation

$$\begin{aligned}\dot{u}(t)&=D(t)u(t),\\ u(0)&=Su(T).\end{aligned}$$

To obtain the Hill-type formula of the above system, we will consider the following Hamiltonian system,

$$\dot{z}(t)=JB(t)z(t), \tag{3.7}$$
$$z(0)=\bar{S}z(T), \tag{3.8}$$

where $B(t)=V\begin{pmatrix}iD & \\ & 0_n\end{pmatrix}V^{-1}$, $\bar{S}=\begin{pmatrix}S & \\ & S\end{pmatrix}$. Changing the basis by V, we have

$$V^*\Big(-J\frac{d}{dt}\Big)V=\begin{pmatrix}i\frac{d}{dt} & \\ & -i\frac{d}{dt}\end{pmatrix},\ V^*B(t)V=\begin{pmatrix}iD & \\ & 0_n\end{pmatrix}\text{ and }V^*JV=\begin{pmatrix}-iI_n & \\ & iI_n\end{pmatrix}.$$

Let $\gamma(t)$ be the fundamental solution of (3.7). Under the new basis, it is obvious that

$$\gamma(t)=\begin{pmatrix}\gamma_D(t) & \\ & I_n\end{pmatrix},$$

where $\gamma_D(t)$ satisfied $\dot{\gamma}_D(t)=D(t)\gamma_D(t)$ with $\gamma_D(0)=I_n$.

By the Hill-type formula (2.7) for Hamiltonian system, we have

$$\begin{aligned}&\det\left[\left(\begin{pmatrix}i\frac{d}{dt} & \\ & -i\frac{d}{dt}\end{pmatrix}-\begin{pmatrix}iD & \\ & 0_n\end{pmatrix}-\begin{pmatrix}-i\nu I_n & \\ & i\nu I_n\end{pmatrix}\right)\begin{pmatrix}i\frac{d}{dt}+P_0 & \\ & -i\frac{d}{dt}+P_0\end{pmatrix}^{-1}\right]\\ &=C(\bar{S})e^{-n\nu T}\exp\left[-\frac{1}{2}\int_0^T Tr\begin{pmatrix}D & \\ & 0_n\end{pmatrix}dt\right]\det\left(\bar{S}\gamma(T)-e^{\nu T}I_{2n}\right).\end{aligned}$$

By Proposition 2.1(2), we can rewrite the above equation as

$$\begin{aligned}&\det\left[\Big(i\frac{d}{dt}-iD+i\nu I_n\Big)\Big(i\frac{d}{dt}+\hat{P}_0\Big)^{-1}\right]\cdot\det\left[\Big(-i\frac{d}{dt}-i\nu I_n\Big)\Big(-i\frac{d}{dt}+\hat{P}_0\Big)^{-1}\right]\\ &=C(\bar{S})e^{-n\nu T}\exp\left[-\frac{1}{2}\int_0^T Tr(D)dt\right]\det\left(S\gamma_D(T)-e^{\nu T}I_n\right)\det\left(S-e^{\nu T}I_n\right).\end{aligned}\tag{3.9}$$

To calculate the determinant $\det\left[\left(i\frac{d}{dt}-iD+i\nu I_n\right)\left(i\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]$, it suffices to calculate the determinant $\det\left[\left(-i\frac{d}{dt}-i\nu I_n\right)\left(-i\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]$. Let $S_1=S|_W$, which is an orthogonal matrix on $\mathbb{R}^{n-k_0}$, obviously $S=\begin{pmatrix} I_{k_0} & \\ & S_1\end{pmatrix}$. We only need to compute $\det\left[\left(-i\frac{d}{dt}-i\nu I_{k_0}\right)\left(-i\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]$ with T-periodic boundary condition and $\det\left[\left(-i\frac{d}{dt}-i\nu I_{n-k_0}\right)\left(-i\frac{d}{dt}\right)^{-1}\right]$ with S_1 boundary condition respectively. Firstly, notice that the spectrum of $i\frac{d}{dt}$ with T-periodic boundary condition

$$\sigma\Big(i\frac{d}{dt}\Big)=\left\{\frac{2k\pi}{T},k\in\mathbb{Z}\right\}.$$

Moreover, by direct computation

$$\begin{aligned}\prod_{k\in\mathbb{Z}}\Big(1+\frac{i\nu}{2k\pi/T}\Big) &= i\nu\prod_{k\in\mathbb{N}}\Big(1+\frac{\nu^2}{(2k\pi/T)^2}\Big)\\ &= \frac{2i}{T}\sinh(T\nu/2)\\ &= \frac{i}{T}e^{-T\nu/2}(e^{T\nu}-1).\end{aligned}$$

We have

$$\det\left[\left(-i\frac{d}{dt}-i\nu I_{k_0}\right)\left(-i\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]=(-1)^{k_0}i^{k_0}\frac{1}{T^{k_0}}e^{-Tk_0\nu/2}\det(I_{k_0}-e^{T\nu}). \tag{3.10}$$

Secondly, writing $\bar{S}_1=\begin{pmatrix} S_1 & \\ & S_1\end{pmatrix}$, by the Hill-type formula for Hamiltonian system with $\bar{S}_1$ boundary condition, we have

$$\det\left[\begin{pmatrix} i\frac{d}{dt}+i\nu & \\ & -i\frac{d}{dt}-i\nu\end{pmatrix}\begin{pmatrix} i\frac{d}{dt} & \\ & -i\frac{d}{dt}\end{pmatrix}^{-1}\right]=C(\bar{S}_1)e^{-(n-k_0)\nu T}\det\left(\bar{S}_1-e^{\nu T}I_{n-k_0}\right)^2.$$

Therefore

$$\det\left[\left(-i\frac{d}{dt}-i\nu I_{n-k_0}\right)\left(-i\frac{d}{dt}\right)^{-1}\right]=\sigma C(\bar{S}_1)^{\frac12}e^{-\frac{(n-k_0)\nu T}{2}}\det\left(S_1-e^{\nu T}I_{n-k_0}\right), \tag{3.11}$$

where σ can be ±1, to be decided. Please note that, when we take $\nu=0$, the left hand side of (3.11) equals to 1 and the right hand side equals to $\sigma\det\left(S_1-I_{n-k_0}\right)C(\bar{S})^{\frac12}$. Since $C(\bar{S})>0$ and σ is a square root of 1, we have

$$\sigma=sign(\det\left(S_1-I_{n-k_0}\right)).$$

Since $C(\bar{S}_1)=\left[\det((S_1-I_{n-k_0})^{-1})\right]^2$, it follows that $\sigma C(\bar{S}_1)^{\frac12}=\det((S_1-I_{n-k_0})^{-1})$, and hence

$$\det\left[\left(-i\frac{d}{dt}-i\nu I_{n-k_0}\right)\left(-i\frac{d}{dt}\right)^{-1}\right]=C(S_1)e^{-\frac{(n-k_0)\nu T}{2}}\det\left(S-e^{\nu T}I_{n-k_0}\right), \tag{3.12}$$

where $C(S_1)=\det((S_1-I_{n-k_0})^{-1})$. From (3.10) and (3.12), we have the following lemma.

Lemma 3.4. *Let S be an orthogonal matrix. Then*

$$\det\left[\left(-i\frac{d}{dt}-i\nu I_n\right)\left(-i\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]=(-1)^{k_0}i^{k_0}C(S)e^{-\frac{n\nu T}{2}}\det\left(S-e^{\nu T}I_n\right), \tag{3.13}$$

where $C(S)=\frac{C(S_1)}{T^{k_0}}$.

Substituting (3.13) in (3.9), we have the following proposition.

Proposition 3.5. *Under the above condition, we have*

$$\det\left[\left(i\frac{d}{dt}-iD+i\nu I_n\right)\left(i\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]=(-1)^{k_0}i^{k_0}C(S)e^{-\frac{n\nu T}{2}}e^{-\frac{1}{2}\int_0^T Tr(D)dt}\det(S\gamma_D(T)-e^{\nu T}I_n).$$

Notice that

$$\begin{aligned}
&\det\left[\left(i\frac{d}{dt}-iD+i\nu I_n\right)\left(i\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]\\
&=\det\left[\left(\frac{d}{dt}-D+\nu I_n\right)\left(\frac{d}{dt}-i\hat{P}_0\right)^{-1}\right]\\
&=\det\left[\left(\frac{d}{dt}-D+\nu I_n\right)\left(\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]\det\left[\left(\frac{d}{dt}+\hat{P}_0\right)\left(\frac{d}{dt}-i\hat{P}_0\right)^{-1}\right]\\
&=\det\left[\left(\frac{d}{dt}-D+\nu I_n\right)\left(\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]i^{k_0},
\end{aligned}$$

where the second equality holds true because of Proposition 2.1(1). Therefore, we have the following theorem which is just Theorem 1.1.

Theorem 3.6. *Under the above assumption,*

$$\begin{aligned}
&\det\left[\left(\frac{d}{dt}-D+\nu I_n\right)\left(\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]\\
&=(-1)^{k_0}C(S)e^{-\frac{n\nu T}{2}}e^{-\frac{1}{2}\int_0^T Tr(D)dt}\det(S\gamma_D(T)-e^{\nu T}I_n).
\end{aligned} \tag{3.14}$$

Please note that

$$(-1)^{k_0}C(S)=(-1)^n|C(S)|,$$

thus we have

$$\begin{aligned}
&\det\left[\left(\frac{d}{dt}-D+\nu I_n\right)\left(\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]\\
&=(-1)^n|C(S)|e^{-\frac{n\nu T}{2}}e^{-\frac{1}{2}\int_0^T Tr(D)dt}\det(S\gamma_D(T)-e^{\nu T}I_n).
\end{aligned}$$

Remark 3.7. *From [5], $\left|\det\Big(\Big(-J\frac{d}{dt}-B\Big)\Big(-J\frac{d}{dt}+P_0\Big)^{-1}\Big)\right|\le G(\|\ B\ \|)$ for some monotone function G. By the above analysis, we have*

$$\det\left[\left(\frac{d}{dt}-D+\nu I_n\right)\left(\frac{d}{dt}+\hat{P}_0\right)^{-1}\right]=\det\Big(\Big(-J\frac{d}{dt}-B\Big)\Big(-J\frac{d}{dt}+P_0\Big)^{-1}\Big)\cdot d(\nu),$$

where $d(\nu)$ is a function depending only on ν and $B(t) = V \begin{pmatrix} iD & \\ & 0_n \end{pmatrix} V^{-1}$. Thus we have that $\det\left[\left(\frac{d}{dt} - \cdot\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right]$ *is a finitely analytic function from $\mathfrak{B}(n)$ to $\mathbb{C}$, and satisfied*

$$\left|\det\left[\left(\frac{d}{dt} - D\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right]\right| \leq G_1(\| D \|),$$

for some monotone function G_1. Moreover, from Theorem 2.4. suppose $\Omega \subseteq \mathbb{C}^m$ be an open set, and $D(Z)$ is analytic map from $\Omega \to \mathfrak{B}(n)$, then $\det\left(\left(\frac{d}{dt} - D(Z)\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right)$ *is an analytic function on Ω.*

By the multiplicative property of conditional Fredholm determinant, Proposition 2.1(1) , we have the following corollary.

Corollary 3.8. *Let $D \in \mathfrak{B}(n)$ such that $\frac{d}{dt} - D$ is invertible, then*

$$\begin{aligned}&\det\left[\left(\frac{d}{dt} - D_1 + \nu I_n\right)\left(\frac{d}{dt} - D\right)^{-1}\right] \\ &= e^{-n\nu T/2} e^{-\frac{1}{2}\int_0^T Tr(D-D_1)dt} \det\left(S\gamma_D(T) - e^{\nu T} I_n\right) \det\left(S\gamma_{D_1} - I_n\right)^{-1}. \end{aligned} \tag{3.15}$$

4 Trace formula for 1st order ODE

In this section, we will derive the trace formula from the Hill-type formula. Firstly, we will study the Taylor expansion for linear parameterized Monodromy matrices, which is similar to the case of Hamiltonian systems [4].

4.1 Taylor expansion for linearly parameterized Monodromy matrices

Let $D_0, D \in \mathfrak{B}(n)$. For $\alpha \in \mathbb{C}$, set $D_\alpha = D_0 + \alpha D$, for $\alpha \in \mathbb{C}$, let γ_α be the corresponding fundamental solutions, that is

$$\dot{\gamma}_\alpha(t) = D_\alpha(t)\gamma_\alpha(t).$$

Fixed $\alpha_0 \in \mathbb{C}$, direct computation shows that

$$\begin{aligned}\frac{d}{dt}(\gamma_{\alpha_0}^{-1}(t)\gamma_\alpha(t)) &= \gamma_{\alpha_0}^{-1}(t)(D_\alpha(t) - D_{\alpha_0}(t))\gamma_\alpha(t) \\ &= (\alpha - \alpha_0)\gamma_{\alpha_0}^{-1}(t)D(t)\gamma_{\alpha_0}(t)\gamma_{\alpha_0}^{-1}(t)\gamma_\alpha(t).\end{aligned}$$

Without loss of generality, assume $\alpha_0 = 0$. In what follows, write

$$\hat{\gamma}_\alpha(t) = \gamma_0^{-1}(t)\gamma_\alpha(t),$$

and

$$\hat{D}(t) = \gamma_0^{-1}(t)D(t)\gamma_0(t),$$

thus

$$\frac{d}{dt}\hat{\gamma}_\alpha(t) = \alpha \hat{D}(t)\hat{\gamma}_\alpha(t). \tag{4.1}$$

To simplify the notation, we use "$^{(k)}$" to denote the k-th derivative on α. Taking derivative on α for both sides of (4.1), we get

$$\frac{d}{dt}\hat{\gamma}_\alpha^{(1)}(t) \quad = \quad \hat{D}(t)\hat{\gamma}_\alpha(t) + \alpha \hat{D}_\alpha(t)\hat{\gamma}_\alpha^{(1)}(t). \tag{4.2}$$

By taking $\alpha = 0$, $\widehat{\gamma}_0(t) \equiv I_n$, we have

$$\hat{\gamma}_0^{(1)}(t) = \int_0^t \hat{D}(s)ds.$$

Now, taking derivative on α for both sides of (4.2), we get

$$\frac{d}{dt}\hat{\gamma}_\alpha^{(2)}(t) \quad = \quad 2\hat{D}(t)\hat{\gamma}_\alpha^{(1)}(t) + \alpha \hat{D}(t)\hat{\gamma}_\alpha^{(2)}(t).$$

Take $\alpha = 0$, and we get

$$\hat{\gamma}_0^{(2)}(t) = 2\int_0^t \hat{D}(s)\hat{\gamma}_0^{(1)}(s)ds.$$

By induction,

$$\frac{d}{dt}\hat{\gamma}_0^{(k)}(t) = k\hat{D}(t)\hat{\gamma}_0^{(k-1)}(t),$$

and

$$\hat{\gamma}_0^{(k)}(t) = k\int_0^t \hat{D}(s)\hat{\gamma}_0^{(k-1)}(s)ds.$$

For $t = T$, by Taylor's formula,

$$\hat{\gamma}_\alpha(T) = I_{2n} + \alpha\hat{\gamma}_0^{(1)}(T) + \cdots + \alpha^k\hat{\gamma}_0^{(k)}(T)/k! + \cdots,$$

where

$$\hat{\gamma}_0^{(1)}(T) = \int_0^T \hat{D}(t)dt$$

and

$$\hat{\gamma}_0^{(k)}(T)/k! = \int_0^T \hat{D}(t)\hat{\gamma}_0^{(k-1)}(t)/(k-1)!dt, k \in \mathbb{N}.$$

By induction, we have

$$\hat{\gamma}_0^{(k)}(T)/k! = \int_0^T \hat{D}(t_1)\int_0^{t_1} \hat{D}(t_2)\cdots\int_0^{t_{k-1}} \hat{D}(t_k)dt_k\cdots dt_2 dt_1, k \in \mathbb{N}.$$

Obviously $\hat{\gamma}_\alpha(T)$ is an entire function on the variable α. We summarize the above reasoning as the following proposition.

Proposition 4.1. *Let $D_\alpha = D_0 + \alpha D$, $\gamma_\alpha(T)$ be the corresponding fundamental solutions. Write $\hat{\gamma}_\alpha = \gamma_0^{-1}\gamma_\alpha$. Then, the Taylor expansion for $\hat{\gamma}_\alpha(T)$ at 0 is*

$$\hat{\gamma}_\alpha(T) = I_n + \alpha\hat{\gamma}_0^{(1)}(T) + \cdots + \alpha^k\hat{\gamma}_0^{(k)}(T)/k! + \cdots , \tag{4.3}$$

where

$$\hat{\gamma}_0^{(k)}(T)/k! = \int_0^T \hat{D}(t_1)\int_0^{t_1}\hat{D}(t_2)\cdots\int_0^{t_{k-1}}\hat{D}(t_k)dt_k\cdots dt_2dt_1, k \in \mathbb{N}. \tag{4.4}$$

In what follows, to simplify the notation, set

$$M(\alpha) = \hat{\gamma}_\alpha(T), \quad M_0 = I_n \quad \text{and} \quad M_j = \hat{\gamma}_0^{(j)}(T)/j!,\ j \in \mathbb{N},$$

then

$$M(\alpha) = \sum_{j=0}^{\infty}\alpha^j M_j.$$

Set $M = S\gamma_0(T)$, then $S\gamma_\alpha(T) = MM(\alpha)$. For $\lambda \in \mathbb{C}$, which is not an eigenvalue of M, by some easy computations, we have that

$$\begin{aligned}\det(S\gamma_\alpha(T) - \lambda I_{2n}) &= \det(MM(\alpha) - \lambda I_{2n})\\ &= \det(M - \lambda I_n + \alpha MM_1 + \cdots + \alpha^k MM_k + \cdots)\\ &= \det(M - \lambda I_n)\det(I_n + \cdots + \alpha^k(M - \lambda I_n)^{-1}MM_k + \cdots).\end{aligned}$$

Let

$$G_k = (M - \lambda I_n)^{-1}MM_k, \tag{4.5}$$

and

$$f(\alpha) = \det(I_n + \cdots + \alpha^k G_k + \cdots),$$

which is an analytic function on $\mathbb{C}$. Next, we will compute the Taylor expansion for $f(\alpha)$. Let $G(\alpha) = \sum_{k=1}^{\infty}\alpha^{k-1}G_k$, then for α small enough, by Theorem 2.6, we have

$$\begin{aligned}f(\alpha) &= \det(I_n + \alpha G(\alpha))\\ &= \exp\Big(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\alpha^m Tr\big(G(\alpha)^m\big)\Big)\\ &= \exp\Big(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\alpha^m Tr\Big[\Big(\sum_{k=1}^{\infty}\alpha^{k-1}G_k\Big)^m\Big]\Big)\\ &= \exp\Big(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\Big[\sum_{k_1,\cdots,k_m=1}^{\infty}\alpha^{k_1+\cdots+k_m}Tr(G_{k_1}\cdots G_{k_m})\Big]\Big).\end{aligned} \tag{4.6}$$

Since $f(\alpha)$ vanishes nowhere near 0, we can write $f(\alpha) = e^{g(\alpha)}$, then by (4.6), some direct computation shows that

$$g^{(m)}(0)/m! = \sum_{k=1}^{m}\frac{(-1)^{k+1}}{k}\Big(\sum_{j_1+\cdots+j_k=m}Tr(G_{j_1}\cdots G_{j_k})\Big). \tag{4.7}$$

For α small enough, let $g(\alpha)$ be the function satisfying

$$\det(S\gamma_\alpha(T)-\lambda I_n)=\det(M-\lambda I_n)\cdot\exp(g(\alpha)), \tag{4.8}$$

then the coefficients $g^{(k)}(0)/k!$ could be determined by (4.7). And we have the following theorem, which is the main result in this subsection.

Theorem 4.2. *Under the above assumption, let $g(\alpha)$ be the function in (4.8). Let $g(\alpha)=\sum\limits_{m=1}^{\infty}c_m\alpha^m$ be its Taylor expansion. Then*

$$c_m=\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k}\Big(\sum_{j_1+\cdots+j_k=m}Tr(G_{j_1}\cdots G_{j_k})\Big). \tag{4.9}$$

where $G_k=(M-\lambda I_n)^{-1}MM_k$, and $M=S\gamma_0(T)$,

$$M_k=\int_0^T\hat{D}(t_1)\int_0^{t_1}\hat{D}(t_2)\cdots\int_0^{t_{k-1}}\hat{D}(t_k)dt_k\cdots dt_2dt_1, k\in\mathbb{N}. \tag{4.10}$$

We only list the first 4 terms

$$g^{(1)}(0)=Tr(G_1),$$

$$g^{(2)}(0)/2=Tr(G_2)-\frac{1}{2}Tr(G_1^2),$$

$$g^{(3)}(0)/3!=Tr(G_3)-Tr(G_1G_2)+\frac{1}{3}Tr(G_1^3),$$

$$g^{(4)}(0)/4!=Tr(G_4)-\frac{1}{2}Tr(G_2^2)-Tr(G_1G_3)+Tr(G_1^2G_2)-\frac{1}{4}Tr(G_4).$$

By the definition of G_k,

$$Tr(G_1)=Tr(M_1M(M-\lambda I_{2n})^{-1})=Tr\Big(\int_0^T\hat{D}(s)ds\cdot M(M-\lambda I_{2n})^{-1}\Big),$$

$$Tr(G_2)=Tr(M_2M(M-\lambda I_{2n})^{-1})=Tr\Big(\int_0^T\hat{D}(s)\int_0^s\hat{D}(\sigma)d\sigma ds\cdot M(M-\lambda I_{2n})^{-1}\Big).$$

Generally,

$$Tr(G_k^m)=Tr\Big(\Big[\int_0^T\hat{D}(t_1)\int_0^{t_1}\hat{D}(t_2)\cdots\int_0^{t_{k-1}}\hat{D}(t_k)dt_k\cdots dt_2dt_1\cdot M(M-\lambda I_{2n})^{-1}\Big]^m\Big),$$

and $Tr(G_{j_1}\cdots G_{j_k})$ could be given similarly.

In the case of Hamiltonian, there are some symmetry property of M_k, please refer [4] for the detail.

4.2 Trace formula for linearly parameterized first order ODE

For $D_\alpha = D_0 + \alpha D$, let D_α take place of D in both sides of Hill-type formula (1.4), we get

$$\begin{aligned} &\det\left[\left(\frac{d}{dt} - D_\alpha + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right] \\ &= (-1)^n |C(S)| e^{-\frac{n\nu T}{2}} e^{-\frac{1}{2}\int_0^T Tr(D_\alpha)dt} \det(S\gamma_{D_\alpha}(T) - e^{\nu T} I_n). \end{aligned} \tag{4.11}$$

As we have proved that, both sides of (4.11) are analytic functions on α. Notice that the left hand side

$$\begin{aligned} \left(\frac{d}{dt} - D_\alpha + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1} &= \left(\frac{d}{dt} - D_0 + \alpha D + \nu I_n\right)\left(\frac{d}{dt} - D_0 + \nu I_n\right)^{-1} \\ &\quad \cdot \left(\frac{d}{dt} - D_0 + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}. \end{aligned} \tag{4.12}$$

Hence

$$\begin{aligned} \det\left[\left(\frac{d}{dt} - D_\alpha + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right] &= \det\left[\left(\frac{d}{dt} - D_0 - \alpha D + \nu I_n\right)\left(\frac{d}{dt} - D_0 + \nu I_n\right)^{-1}\right] \\ &\quad \cdot \det\left[\left(\frac{d}{dt} - D_0 + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right] \\ &= \det\left(id - \alpha D\left(\frac{d}{dt} - D_0 + \nu I_n\right)^{-1}\right) \\ &\quad \cdot \det\left[\left(\frac{d}{dt} - D_0 + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right] \end{aligned} \tag{4.13}$$

Let

$$F = D\left(\frac{d}{dt} - D_0 + \nu I_n\right)^{-1},$$

then the left hand side of (4.11)

$$\begin{aligned} f(\alpha) &= \det\left[\left(\frac{d}{dt} - D_\alpha + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right] \\ &= \det\left[\left(\frac{d}{dt} - D_\alpha + \nu I_n\right)\left(\frac{d}{dt} - D_0 + \nu I_n\right)^{-1}\right] \det\left[\left(\frac{d}{dt} - D_0 + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right] \\ &= \det(id - \alpha F) \cdot \det\left[\left(\frac{d}{dt} - D_0 + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right]. \end{aligned} \tag{4.14}$$

Notice that F is a Hilbert-Schmidt operator with trace finite condition with respect to $\hat{P}_k$. By Theorem 2.6, for α small,

$$f(\alpha) = \exp\left(\sum_{m=1} \frac{-1}{m}\alpha^m Tr(F^m)\right) \cdot \det\left[\left(\frac{d}{dt} - D_0 + \nu I_n\right)\left(\frac{d}{dt} + \hat{P}_0\right)^{-1}\right]. \tag{4.15}$$

On the other hand, by Theorem 4.2, the right hand side of (4.11) equals to

$$
\begin{aligned}
&(-1)^n|C(S)|e^{-\frac{n\nu T}{2}}e^{-\frac{1}{2}\int_0^T Tr(D_\alpha)dt}\det(S\gamma_\alpha(T)-e^{\nu T}I_n)\\
&=(-1)^n|C(S)|e^{-\frac{n\nu T}{2}}e^{-\frac{1}{2}\int_0^T Tr(D_0)dt}\det\left(M-\lambda I_n\right)e^{-\frac{\alpha}{2}\int_0^T Tr(D)dt}e^{g(\alpha)},
\end{aligned}
\tag{4.16}
$$

where $g(\alpha)=\sum\limits_{m=1}^{\infty}c_m\alpha^m$ satisfies

$$
\det(S\gamma_\alpha(T)-e^{\nu T}I_n)=\det(M-\lambda I_n)e^{g(\alpha)},
$$

and c_m are given in (4.9). Comparing (4.15) with (4.16), we have

$$
g(\alpha)=\frac{\alpha}{2}\int_0^T Tr(D)dt+\sum_{m=1}\frac{-1}{m}\alpha^m Tr(F^m),
$$

and hence

$$
Tr(F)=\frac{1}{2}\int_0^T Tr(D(t))dt-c_1,
\tag{4.17}
$$

and

$$
Tr(F^m)=-mc_m,\ \ m\geq 2.
\tag{4.18}
$$

Thus we get

Theorem 4.3. *Let $\nu\in\mathbb{C}$ such that $\frac{d}{dt}-D_0-\nu$ is invertible, $F=D\left(\frac{d}{dt}-D_0+\nu I_n\right)^{-1}$, then*

$$
Tr(F)=\frac{1}{2}\int_0^T Tr(D(t))dt-Tr(G_1),
\tag{4.19}
$$

and for any positive integer $m\geq 2$,

$$
Tr(F^m)=m\sum_{k=1}^{m}\frac{(-1)^k}{k}\Big[\sum_{j_1+\cdots+j_k=m}Tr(G_{j_1}\cdots G_{j_k})\Big].
\tag{4.20}
$$

where $G_k=(M-\lambda I_{2n})^{-1}MM_k$, and $M=S\gamma_0(T)$,

$$
M_k=\int_0^T\hat{D}(t_1)\int_0^{t_1}\hat{D}(t_2)\cdots\int_0^{t_{k-1}}\hat{D}(t_k)dt_k\cdots dt_2dt_1, k\in\mathbb{N}.
\tag{4.21}
$$

For large m, the right hand side of (4.20) is a little complicated. However, for $m=1,2$, we can write it down more precisely.

Corollary 4.4. *Under the assumption as in Theorem 4.3,*

$$
Tr(F)=\frac{1}{2}\int_0^T Tr(D(t))dt-Tr\Big(\int_0^T\gamma_0^{-1}(t)D(t)\gamma_0(t)dt\cdot M(M-e^{\nu T}I_n)^{-1}\Big).
$$

and

$$
\begin{aligned}
Tr(F^2)\ \ =\ \ &-2Tr\Big(\int_0^T\gamma_0^{-1}(t)D(t)\gamma_0(t)\int_0^t\gamma_0^{-1}(s)D(s)\gamma_0(s)dsdt\cdot M(M-\lambda I_n)^{-1}\Big)\\
&+Tr\Big(\Big[\int_0^T\gamma_0^{-1}(t)D(t)\gamma_0(t)dt\cdot M(M-e^{\nu T}I_n)^{-1}\Big]^2\Big).
\end{aligned}
$$

Especially

$$Tr\left(D\left(\frac{d}{dt}+\nu I_n\right)^{-1}\right) = Tr\Big(\int_0^T D(t)dt\cdot(\frac{1}{2}-S(S-e^{\nu T}I_n)^{-1})\Big)$$
$$= -\frac{1}{2}Tr\Big(\int_0^T D(t)dt\cdot(S+e^{\nu T})(S-e^{\nu T}I_n)^{-1})\Big). \qquad (4.22)$$

Taking derivative on both sides of (4.22), we get

$$Tr\left(D\left(\frac{d}{dt}+\nu I_n\right)^{-2}\right) = -Te^{\nu T}Tr\Big(\int_0^T D(t)dt\cdot S(S-e^{\nu T}I_n)^{-2})\Big). \qquad (4.23)$$

5 Examples

Krein considered second order system in [6]

$$y''+\lambda R(t)y=0,\ \ y(0)+y(T)=y'(0)+y'(T)=0, \qquad (5.1)$$

where $R(t)$ is continuous path of real symmetric matrices on $\mathbb{R}^n$. Set $R_{ave}=\frac{1}{T}\int_0^T R(t)dt$ and

$$X(t)=\int_0^t (R(s)-R_{ave})ds+C, \qquad (5.2)$$

where C is a constant matrix which is chosen such that $X_{ave}=0$. Let λ_j be the eigenvalues of (5.1), Krein get

$$\sum\frac{1}{\lambda_j}=\frac{T}{4}\int_0^T Tr(R(t))dt, \qquad (5.3)$$

and

$$\sum\frac{1}{\lambda_j^2}=\frac{T}{2}\int_0^T Tr(X^2(t))dt+\frac{T^2}{48}Tr[(\int_0^T R(t)dt)^2]. \qquad (5.4)$$

In this section, we will give a generalization of Krein's trace formula from our viewpoint. To simplify the notation, let $\mathcal{A}(\nu)=-(\frac{d}{dt}+\nu)^2$ and denote by

$$R_{ave}=\frac{1}{T}\int_0^T R(t)dt,$$

which is a constant matrix. From (4.23), we have

$$Tr(R\mathcal{A}(\nu)^{-1})=-\omega T^2\cdot Tr(R_{ave}\cdot S(S-\omega)^{-2}), \qquad (5.5)$$

By taking derivative with respect to ν on both sides of (5.5), we get

$$Tr(R\mathcal{A}(\nu)^{-2})=\frac{\omega T^4}{6}Tr(R_{ave}S(S^2+4\omega S+\omega^2)(S-\omega)^{-4}). \qquad (5.6)$$

Especially, if $\int_0^T Rdt=0$, then

$$Tr(R\mathcal{A}(\nu)^{-2})=0. \qquad (5.7)$$

Please note that (5.5) is obtained by taking derivative on both sides of (4.22). Obviously, in this case, we could calculate $Tr\Big(D\Big(\frac{d}{dt}+\nu I_n\Big)^{-k}\Big)$ for any $k\in\mathbb{N}$ by taking derivative with respect to ν for both sides of (4.22).

The purpose of the remaining part of this subsection is to compute $Tr\Big[\Big(R\mathcal{A}(\nu)^{-1}\Big)^2\Big]$. In fact, we will use our trace formula (4.20) to express $Tr\Big[\Big(R\mathcal{A}(\nu)^{-1}\Big)^2\Big]$ as the form of multiple integral. Inspired by Krein [6], let

$$X(t)=\int_0^t (R(s)-R_{ave})ds+C,$$

where C is a constant matrix. Obviously,

$$X(0)=X(T)=C.$$

At first, we will calculate $Tr\Big[\Big((R-R_{ave})\mathcal{A}(\nu)^{-1}\Big)^2\Big]$.

Proposition 5.1. *For any constant matrix C which satisfies $CS=SC$, $X(t)$ is defined as above, then*

$$\begin{aligned}&Tr\Big[\Big((R-R_{ave})\mathcal{A}(\nu)^{-1}\Big)^2\Big]\\&=-2\omega Tr\Big[T\int_0^T X^2dt\cdot S(S-\omega)^{-2}\Big]+2Tr\Big[\Big(\int_0^T X(t)dt\cdot S(S-\omega)^{-1}\Big)^2\Big]\\&\quad -4Tr\Big[\int_0^T X(t)\int_0^t X(s)dsdt\cdot S(S-\omega)^{-1}\Big].\end{aligned}\tag{5.8}$$

Proof. Recall that $\mathcal{A}(\nu)=-(\frac{d}{dt}+\nu)^2$. Please note that for any bounded operators A,B such that AB is trace class operator, we have that $Tr(AB)=Tr(BA)$. Hence

$$\begin{aligned}Tr\Big[\Big((R-R_{ave})\mathcal{A}(\nu)^{-1}\Big)^2\Big]&=Tr\Big[(R-R_{ave})\Big(\frac{d}{dt}+\nu\Big)^{-2}(R-R_{ave})\Big(\frac{d}{dt}+\nu\Big)^{-2}\Big]\\&=Tr\Big[\Big(\Big(\frac{d}{dt}+\nu\Big)^{-1}(R-R_{ave})\Big(\frac{d}{dt}+\nu\Big)^{-1}\Big)^2\Big].\end{aligned}\tag{5.9}$$

Noting that $X(t)=\int_0^t(R(s)-R_{ave})ds+C$ for some constant matrix C, we have

$$\Big(\frac{d}{dt}+\nu\Big)X-X\Big(\frac{d}{dt}+\nu\Big)=\dot{X}=R-R_{ave},\tag{5.10}$$

thus

$$X\Big(\frac{d}{dt}+\nu\Big)^{-1}-\Big(\frac{d}{dt}+\nu\Big)^{-1}X=\Big(\frac{d}{dt}+\nu\Big)^{-1}(R-R_{ave})\Big(\frac{d}{dt}+\nu\Big)^{-1}.\tag{5.11}$$

By (5.9) and (5.11),

$$\begin{aligned}Tr\Big[\Big((R-R_{ave})\mathcal{A}(\nu)^{-1}\Big)^2\Big]&=Tr\Big[\Big(X\Big(\frac{d}{dt}+\nu\Big)^{-1}-\Big(\frac{d}{dt}+\nu\Big)^{-1}X\Big)^2\Big]\\&=2Tr\Big[\Big(X\Big(\frac{d}{dt}+\nu\Big)^{-1}\Big)^2\Big]-2Tr\Big[X^2\Big(\frac{d}{dt}+\nu\Big)^{-2}\Big]\\&=2Tr\Big[\Big(X\Big(\frac{d}{dt}+\nu\Big)^{-1}\Big)^2\Big]+2Tr\left[X^2\mathcal{A}(\nu)^{-1}\right].\end{aligned}\tag{5.12}$$

From (5.5)

$$Tr\Big[X^2\mathcal{A}(\nu)^{-1}\Big] = -\omega T Tr\left[\int_0^T X^2 dt \cdot S(S-\omega)^{-2}\right]. \tag{5.13}$$

To continue, we should calculate $Tr\left[\left(X\left(\frac{d}{dt}+\nu\right)^{-1}\right)^2\right]$ by using Corollary 4.4. In this case, D_0 in Corollary 4.4 is 0, thus $\gamma_0(t) = I_n$, $M = S$. It follows that

$$\begin{aligned} & Tr\left[\left(X\left(\frac{d}{dt}+\nu\right)^{-1}\right)^2\right] \\ & = Tr\left[\left(\int_0^T X(t)dt \cdot S(S-\omega)^{-1}\right)^2\right] - 2Tr\left[\int_0^T X(t)\int_0^t X(s)dsdt \cdot S(S-\omega)^{-1}\right]. \end{aligned} \tag{5.14}$$

By substituting (5.14) and (5.13) into (5.12), we have the desired result. □

Please note that if $S = \pm I_n$, then we have

$$Tr\left[\int_0^T X(t)\int_0^t X(s)dsdt \cdot S(S-\omega)^{-1}\right] = \frac{1}{2}Tr\left[\left(\int_0^T X(t)dt\right)^2 \cdot S(S-\omega)^{-1}\right] \tag{5.15}$$

Moreover, in this case, the constant matrix C in Proposition 5.1 could be chosen arbitrary. Particularly, C could be chosen such that $\int_0^T X(t)dt = 0$.

Corollary 5.2. *In the case $S = \pm I_n$, if we choose C such that $\int_0^T X(t)dt = 0$, then we have*

$$Tr\left[\left((R-R_{ave})\mathcal{A}(\nu)^{-1}\right)^2\right] = -\frac{2\omega}{(1\mp\omega)^2}Tr\left[T^2\int_0^T X^2 dt\right]. \tag{5.16}$$

Proposition 5.3. *Suppose $R_{ave}S = SR_{ave}$, then*

$$\begin{aligned} & Tr((R\mathcal{A}(\nu)^{-1})^2) \\ & = \frac{\omega T^4}{6}Tr(R_{ave}^2 S(S^2+4\omega S+\omega^2)(S-\omega)^{-4}) - 2\omega T\cdot Tr\left(\int_0^T X^2 dt\cdot S(S-\omega)^{-2}\right) \\ & \quad +2Tr\left(\left(\int_0^T X(t)dt\cdot S(S-\omega)^{-1}\right)^2\right) - 4Tr\left(\int_0^T X(t)\int_0^t X(s)ds\cdot S(S-\omega)^{-1}\right) \end{aligned} \tag{5.17}$$

Proof.

$$\begin{aligned} \left(R\mathcal{A}(\nu)^{-1}\right)^2 &= ((R_{ave}+(R-R_{ave}))\mathcal{A}(\nu)^{-1})^2 \\ &= \left((R-R_{ave})\mathcal{A}(\nu)^{-1}\right)^2 + \left(R_{ave}\mathcal{A}(\nu)^{-1}\right)^2 \\ & \quad +(R-R_{ave})\mathcal{A}(\nu)^{-1}R_{ave}\mathcal{A}(\nu)^{-1} + R_{ave}\mathcal{A}(\nu)^{-1}(R-R_{ave})\mathcal{A}(\nu)^{-1}. \end{aligned} \tag{5.18}$$

Please note that $R_{ave}S = SR_{ave}$ implies that $\mathcal{A}(\nu)^{-1}$ commutes with R_{ave}. We have

$$\begin{aligned} Tr(R\mathcal{A}(\nu)^{-1})^2) &= Tr(((R-R_{ave})\mathcal{A}(\nu)^{-1})^2) + Tr(R_{ave}\mathcal{A}(\nu)^{-1})^2) \\ & \quad +2Tr((R-R_{ave})R_{ave}\mathcal{A}(\nu)^{-2}). \end{aligned} \tag{5.19}$$

Direct computation shows that

$$\begin{aligned} Tr(R_{ave}\mathcal{A}(\nu)^{-1})^2) &= Tr(R_{ave}^2\mathcal{A}(\nu)^{-2}) \\ &= \frac{\omega T^4}{6} Tr(R_{ave}^2 S(S^2+4\omega S+\omega^2)(S-\omega)^{-4}). \end{aligned} \tag{5.20}$$

Since $\int_0^T (R-R_{ave})dt = 0$, by (5.7) we have

$$Tr\Big[(R-R_{ave})R_{ave}\mathcal{A}(\nu)^{-2}\Big] = 0. \tag{5.21}$$

Combining (5.19) with (5.8), (5.20) and (5.21), the desired result is proved. □

Corollary 5.4. *In the case $S = \pm I_n$, if we choose C such that $\int_0^T X(t)dt = 0$, then we have*

$$\begin{aligned} &Tr((R\mathcal{A}(\nu)^{-1})^2) \\ &= \frac{\pm(1\pm 4\omega+\omega^2)\omega T^2}{6(1\mp\omega)^4} Tr\left[\left(\int_0^T R(t)dt\right)^2\right] \mp \frac{2\omega T}{(1\mp\omega)^2}\cdot\left(\int_0^T Tr(X^2)dt\right). \end{aligned} \tag{5.22}$$

Remark 5.5. *More specially, for the case that Krein considered, that is, let $S = -I_n$ and $\nu = 0$, in this case, $\omega = 1$. By (5.5), we have Krein's trace formula (5.3). Moreover, by (5.22), we have*

$$\sum \frac{1}{\lambda_j^2} = \frac{T}{2}\int_0^T Tr(X^2(t))dt + \frac{T^2}{48} Tr\Big[\Big(\int_0^T R(t)dt\Big)^2\Big],$$

which is Krein's trace formula (5.4).

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